

The Improved Preconditioned Accelerated Overrelaxation (AOR) Iterative Method for L –Matrix

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ABSTRACT

Many authors have presented preconditioned schemes for linear system $Ax=b$ to improve the convergence rates of Accelerated overrelaxation AOR (Successive Overrelaxation) iterative schemes. The preconditioner that strike a balance between the two general preconditioners are proposed, in this present work. Some comparison theorems on preconditioned iterative schemes for solving L-Matrix linear system are analyzed. From comparison theorems, we can conclude that the present preconditioner can apply to improve the convergence of AOR iterative scheme under mild assumption on coefficient matrix A . Some numerical examples are presented to support the theoretical results of this paper.

Keywords: Accelerated Overrelaxation Method, Preconditioner, Convergence, L – matrix, spectral radius

INTRODUCTION

For many years many researchers have been studying iterative scheme which approximate the solution of the linear system

$$Ax = b, \tag{1}$$

where $A = (a_{ij}) \in \mathbb{R}^{n \times n}$ is a nonsingular matrix with nonvanishing diagonal elements, and where $x \in \mathbb{R}^n$ and $b \in \mathbb{R}^n$ are respectively vectors of unknown and preassigned variables. Some methods of preconditioning which improve the rate of convergence of these iterative methods have been developed. Recall the usual splitting of the coefficient matrix $A \in \mathbb{R}^{n \times n}$,

$$A = D_A - L_A - U_A \tag{2}$$

where $D_A = \text{diag}(a_{11}, a_{22}, \dots, a_{nn})$ is the diagonal part of A , $-L_A$ and $-U_A$ its strictly lower and strictly upper parts, respectively. If $a_{ii} \neq 0$ for all $i \in \mathbb{N}$ ($i = 1, 2, \dots, n$), then we can multiply the linear system (1) by D_A^{-1} , arising therefrom the splitting of the matrix.

$$A = I - L - U \tag{3}$$

where $I = D_A^{-1}D_A$, $L = D_A^{-1}L_A$ and $U = D_A^{-1}U_A$.

Suppose $A = M - N$ is a regular splitting of the coefficient matrix $A = (a_{ij})$, then the basic iterative method for the solution of system (1) can be expressed in the form

$$\begin{aligned} x^{(k+1)} &= M^{-1}Nx^{(k)} + M^{-1}b, \\ k &= 0, 1, 2, \dots \end{aligned} \tag{4}$$

where $M^{-1}N$ is known as the iteration matrix of the method. The iteration (4) is known to converge to the exact solution $x = A^{-1}b$ for any initial vector value $x^{(0)} \in \mathbb{R}^n$ if and only if the spectral radius $\rho(M^{-1}N) < 1$. The smaller the spectral radius the faster the convergence speed of the iterative method. The classical AOR iterative method given in Hadjidimos (1978) for solving (1) is defined as

$$\begin{aligned} x^{(k+1)} &= \ell_{r,\omega}x^{(k)} + (I - rL)^{-1}\omega b \\ k &= 0, 1, 2, \dots \end{aligned} \tag{5}$$

with the iteration matrix $\mathcal{L}_{r,\omega}$ given as

$$\begin{aligned} \ell_{r,\omega} &= (I - rL)^{-1}[(1 - \omega)I + (\omega - r)L \\ &\quad + \omega U] \end{aligned} \tag{6}$$

where ω and r are real parameters with $\omega \neq 0$. Besides the iteration matrix and parameters involved in the iterative methods, convergence is also dependent on the nature of the linear systems of equations themselves. Therefore, in order to improve efficiency of the iterative method (4), the linear system (1) is transformed into the equivalent preconditioned system

$$PAx = Pb \tag{7}$$

where P is a nonsingular matrix called a preconditioner. Different preconditioners have been advanced by several researchers for the preconditioned system (7). Gunawardena *et al.*

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(1991), Kohno *et al.* (1997), Evans *et al.* (2001), Li *et al.* (2000), Li *et al.* (2007), Wu *et al.* (2007), Wu and Huang (2007), Yun and Kim (2008), Wang and Song (2009), Darvishi *et al.* (2011), Li (2011), Ndanusa and Adeboye (2012), Huang *et al.* (2016), Behzadi (2019) and Wang (2019) are some instances of application of the preconditioned system (7) to improve the convergence of the AOR method. The preconditioner P of Faruk and Ndanusa (2019) is described by $P = I + S'$, where.

$$S' = \begin{bmatrix} 0 & -a_{12} & 0 & \cdots & 0 \\ -a_{21} & 0 & -a_{23} & \cdots & 0 \\ 0 & -a_{32} & 0 & \cdots & \vdots \\ \vdots & \vdots & \ddots & \cdots & -a_{n-1,n} \\ 0 & 0 & 0 & -a_{n,n-1} & 0 \end{bmatrix} \quad (8)$$

$P = I + \bar{S}$ is another preconditioner proposed by Abdullahi and Ndanusa (2020), with

$$\bar{S} = \begin{bmatrix} 0 & -a_{12} & 0 & \cdots & -a_{1n} \\ -a_{21} & 0 & -a_{23} & \cdots & 0 \\ -a_{31} & 0 & 0 & \cdots & \vdots \\ \vdots & \vdots & \cdots & \cdots & -a_{n-1,n} \\ -a_{n1} & 0 & 0 & \cdots & 0 \end{bmatrix} \quad (9)$$

Using the idea of previous works, we consider the preconditioner of the type $P = I + \tilde{S}$, where

$$\tilde{S} = \bar{S} + S' = \begin{bmatrix} 0 & -a_{12} & 0 & \cdots & -a_{1n} \\ -a_{21} & 0 & -a_{23} & \cdots & 0 \\ -a_{31} & -a_{32} & 0 & \cdots & \vdots \\ \vdots & \vdots & \ddots & \cdots & -a_{n-1,n} \\ -a_{n1} & 0 & 0 & -a_{n,n-1} & 0 \end{bmatrix} \quad (10)$$

The present work is aimed at investigating the modified preconditioner $P = I + \tilde{S}$ applied to AOR iteration, through theoretical proofs, corroborated by numerical verifications, in order to improve upon the rate of convergence of the method.

METHODOLOGY

Preliminaries

For convenience, some notations, definitions and lemmas that will be used in the succeeding sections are briefly explained. Suppose $A = (a_{ij}) \in R^{n \times n}$ is a real matrix. The notation $diag(A)$ implies the $n \times n$ diagonal matrix coinciding in its diagonal with a_{ii} . For a square matrix A , $\rho(A)$ denotes the spectral radius of A .

Definition 1 (Young, (1971)). A matrix $A = (a_{ij})_{i,j=1,2,\dots,n}$ is an L -matrix if $a_{ii} > 0$ and $a_{ij} \leq 0$ ($i \neq j$).

Definition 2 (Saad (2000)). An L -matrix $A = (a_{ij})$ where A is nonsingular and $A^{-1} \geq 0$ is called an M -matrix.

Definition 3 (Saad (2000)). A matrix $A = (a_{ij})_{i,j=1,2,\dots,n}$ is called nonnegative, nonpositive and positive if $a_{ij} \geq 0$, $a_{ij} \leq 0$ and $a_{ij} > 0$, respectively.

Definition 4 Dehghan and Hajarian (2009). The decomposition of a real matrix $A \in R^{n \times n}$ into the form $A = M - N$, where M is a nonsingular matrix is called a splitting of A . Such splitting is called

- Regular if $M^{-1} \geq 0$ and $N \geq 0$
- Nonnegative if $M^{-1}N \geq 0$
- Convergent if $\rho(M^{-1}N) < 1$
- M -splitting if M is a nonsingular M -matrix and $N \geq 0$.

Definition 5 (Saad (2000)). $A=(a_{ij})$ is called an irreducible matrix if the directed graph associated to A is strongly connected.

Lemma 1 (Varga (1962)). Let $A \geq 0$ be an irreducible $n \times n$ matrix. Then,

- i. A has a positive real eigenvalue equal to its spectral radius.
- ii. To $\rho(A)$ there corresponds an Eigen vector $x > 0$.
- iii. $\rho(A)$ increases when any entry of A increases.
- iv. $\rho(A)$ is a simple Eigen value of A .

Lemma 2 (Gunawardena *et al.* (1991)). Let A be a nonnegative matrix. Then,

- i. If $\alpha x \leq Ax$ for some nonnegative vector $x, x \neq 0$, then $\alpha \leq \rho(A)$.
- ii. If $Ax \leq \beta x$ for some positive vector x , then $\rho(A) \leq \beta$. Moreover, if A is irreducible and if $0 \neq \alpha x \leq Ax \leq \beta x$ for some nonnegative vector x , then $\alpha \leq \rho(A) \leq \beta$ and x is a positive vector.

Lemma 3 (Varga (1962)). Suppose $A = M - N$ is an M -splitting of A . Then the splitting is convergent iff A is a nonsingular M -matrix.

Lemma 4 (Varga (1962)). Let $A = M_1 - N_1 = M_2 - N_2$ be two regular splittings of A , where $A^{-1} \geq 0$. If $N_2 \geq N_1 \geq 0$, then,

$$1 > \rho(M_2^{-1}N_2) \geq \rho(M_1^{-1}N_1) \geq 0.$$

If moreover, $A^{-1} > O$ and if $N_2 \geq N_1 \geq O$, equality excluded (meaning that neither N_1 nor $N_2 - N_1$ is the null matrix), then $1 > \rho(M_2^{-1}N_2) > \rho(M_1^{-1}N_1) > O$.

The Preconditioned AOR iterative method

Application of the preconditioner P to system (1) results in the preconditioned linear system

$$\tilde{A}x = \tilde{b} \tag{11}$$

where $\tilde{A} = (I + \tilde{S})A$ and $\tilde{b} = (I + \tilde{S})b$ with

$$\tilde{S} = \begin{bmatrix} 0 & -a_{12} & 0 & \cdots & -a_{1n} \\ -a_{21} & 0 & -a_{23} & \cdots & 0 \\ -a_{31} & -a_{32} & 0 & \cdots & \vdots \\ \vdots & \vdots & \ddots & \cdots & -a_{n-1,n} \\ -a_{n1} & 0 & 0 & -a_{n,n-1} & 0 \end{bmatrix}$$

A usual splitting of the preconditioned coefficient matrix \tilde{A} of (11) into its diagonal (\tilde{D}), strictly lower ($-\tilde{L}$) and strictly upper ($-\tilde{U}$) components is obtained thus

$$\tilde{A} = \tilde{D} - \tilde{L} - \tilde{U}$$

with the following resultant representations

$$\begin{aligned} \tilde{D} &= I + D_1, \quad \tilde{L} = L + \tilde{L}_{\tilde{S}} + L_1, \quad \tilde{U} = U + U_{\tilde{S}} + U_1, \\ \tilde{S} &= -L_{\tilde{S}} - U_{\tilde{S}}, \quad \text{and} \quad -\tilde{S}L - \tilde{S}U = D_1 - L_1 - U_1; \end{aligned}$$

where D_1 , $-L_1$, and $-U_1$ are the diagonal, strictly lower and strictly upper parts of $-\tilde{S}L - \tilde{S}U$ respectively; and $-L_{\tilde{S}}$ and $-U_{\tilde{S}}$ are the strictly lower and strictly upper parts of \tilde{S} respectively. Application of the AOR method to the preconditioned linear system (11) results in the corresponding preconditioned AOR method whose iterative matrix is defined by

$$\tilde{\ell}_{r,\omega} = (\tilde{D} - r\tilde{L})^{-1}[(1 - \omega)\tilde{D} + (\omega - r)\tilde{L} + \omega\tilde{U}] \tag{12}$$

Theorem 1 Let $\ell_{r,\omega}$ and $\tilde{\ell}_{r,\omega}$ be the AOR iterative matrices corresponding to the linear systems (1) and (11) respectively. Suppose $0 \leq r \leq \omega \leq 1$ ($\omega \neq 0, r \neq 1$), A is an irreducible L - matrix with $0 < a_{1n}a_{n1} < 1$, $0 < a_{1i}a_{i1} + a_{i,i+1}a_{i+1,i} < 1$ ($i = 2(1)n - 1$) and $0 < a_{12}a_{21} + a_{in}a_{n1} < 1$. Then $\ell_{r,\omega}$ and $\tilde{\ell}_{r,\omega}$ are nonnegative and irreducible matrices.

PROOF: Since A is an L - matrix, $L \geq 0$ and $U \geq 0$. Thus $(I - rL)^{-1} = I + rL + r^2L^2 + \cdots + r^{n-1}L^{n-1} \geq 0$. And, from (6), we have

$$\begin{aligned} \ell_{r,\omega} &= (I - rL)^{-1}[(1 - \omega)I + (\omega - r)L + \omega U] \\ &= [I + rL + r^2L^2 + \cdots + r^{n-1}L^{n-1}][(1 - \omega)I + (\omega - r)L + \omega U] \\ &= (1 - \omega)I + (\omega - r)L + \omega U + rL(1 - \omega)I + rL[(\omega - r)L + \omega U] \\ &\quad + (r^2L^2 + \cdots + r^{n-1}L^{n-1})[(1 - \omega)I + (\omega - r)L + \omega U] \\ &= (1 - \omega)I + [(\omega - r)L + rL(1 - \omega)I] + \omega U + rL[(\omega - r)L + \omega U] \\ &\quad + (r^2L^2 + \cdots + r^{n-1}L^{n-1})[(1 - \omega)I + (\omega - r)L + \omega U] \\ &= (1 - \omega)I + \omega(1 - r)L + \omega U + T \end{aligned}$$

where

$$\begin{aligned} T &= rL[(\omega - r)L + \omega U] \\ &+ (r^2L^2 + \cdots + r^{n-1}L^{n-1}) \times [(1 - \omega)I + (\omega - r)L + \omega U] \geq 0. \end{aligned}$$

It is clear that $(1 - \omega)I + \omega(1 - r)L + \omega U \geq 0$. Consequently, $\ell_{r,\omega} = (1 - \omega)I + \omega(1 - r)L + \omega U + T \geq 0$. Hence, $\ell_{r,\omega}$ is a nonnegative matrix. Since $A = I - L - U$ is irreducible, so also is $(1 - \omega)I + \omega(1 - r)L + \omega U$ since the coefficients of I, L , and U are different from zero and less than 1 in absolute value. Hence, $\ell_{r,\omega}$ is an irreducible matrix.

Now, consider the preconditioned AOR iterative matrix

$$\tilde{\ell}_{r,\omega} = (\tilde{D} - r\tilde{L})^{-1}[(1 - \omega)\tilde{D} + (\omega - r)\tilde{L} + \omega\tilde{U}]$$

Equation (7) ensures that the L - matrix structure of A is preserved in \tilde{A} . Since \tilde{A} is an L - matrix, it is evident that $\tilde{L} \geq 0$ and $\tilde{U} \geq 0$. Also, by the conditions of Theorem 1, it is easy to get that $\tilde{D} \geq 0$. Thus,

$$\begin{aligned} \tilde{\ell}_{r,\omega} &= [\tilde{D}(I - r\tilde{D}^{-1}\tilde{L})]^{-1}[(1 - \omega)\tilde{D} + (\omega - r)\tilde{L} + \omega\tilde{U}] \\ &= (I - r\tilde{D}^{-1}\tilde{L})^{-1}\tilde{D}^{-1}[(1 - \omega)\tilde{D} + (\omega - r)\tilde{L} + \omega\tilde{U}] \\ &= (I - r\tilde{D}^{-1}\tilde{L})^{-1}[(1 - \omega)I + (\omega - r)\tilde{D}^{-1}\tilde{L} + \omega\tilde{D}^{-1}\tilde{U}] \\ &= [I + r\tilde{D}^{-1}\tilde{L} + r^2(\tilde{D}^{-1}\tilde{L})^2 + \cdots + r^{n-1}(\tilde{D}^{-1}\tilde{L})^{n-1}] \end{aligned}$$

$$\begin{aligned} & \times [(1 - \omega)I + (\omega - r)\tilde{D}^{-1}\tilde{L} + \omega\tilde{D}^{-1}\tilde{U}] \\ & = (1 - \omega)I + \omega(1 - r)\tilde{D}^{-1}\tilde{L} + \omega\tilde{D}^{-1}\tilde{U} + \tilde{T} \end{aligned}$$

where

$$\begin{aligned} \tilde{T} & = r\tilde{D}^{-1}\tilde{L}[(\omega - r)\tilde{D}^{-1}\tilde{L} + \omega\tilde{D}^{-1}\tilde{U}] + [r^2(\tilde{D}^{-1}\tilde{L})^2 + \dots + r^{n-1}(\tilde{D}^{-1}\tilde{L})^{n-1}] \\ & \times [(1 - \omega)I + (\omega - r)\tilde{D}^{-1}\tilde{L} + \omega\tilde{D}^{-1}\tilde{U}] \geq 0 \end{aligned}$$

Using similar arguments, it is conclusive that $\tilde{\ell}_{r,\omega}$ is also nonnegative and irreducible. The following equalities are essential to prove Theorem 2.

- A1: $\tilde{U} = U + U_{\tilde{S}} + U_1$
- A2: $\tilde{D} = \bar{D} - S'\bar{S} = I + D_1$
- A3: $\tilde{L} = \bar{L} = L + L_{\tilde{S}} + L_1$
- A4: $\tilde{D} - \tilde{L} = \bar{D} - S'\bar{S} - \bar{L}$

Theorem 2 Let $\ell_{r,\omega} = (I - rL)^{-1}[(1 - \omega)I + (\omega - r)L + \omega U]$ and $\tilde{\ell}_{r,\omega} = (\tilde{D} - r\tilde{L})^{-1}[(1 - \omega)\tilde{D} + (\omega - r)\tilde{L} + \omega\tilde{U}]$ be the AOR and preconditioned AOR iterative matrices respectively. Suppose $0 \leq r \leq \omega \leq 1$ ($\omega \neq 0, r \neq 1$), A is an irreducible $L -$ matrix with $0 < a_{1n}a_{n1} < 1$, $0 < a_{1i}a_{i1} + a_{i,i+1}a_{i+1,i} < 1$ ($i = 2(1)n - 1$) and $0 < a_{12}a_{21} + a_{in}a_{n1} < 1$. Then

- (i) $\rho(\tilde{\ell}_{r,\omega}) < \rho(\ell_{r,\omega})$, if $\rho(\ell_{r,\omega}) < 1$;
- (ii) $\rho(\tilde{\ell}_{r,\omega}) = \rho(\ell_{r,\omega})$, if $\rho(\ell_{r,\omega}) = 1$;
- (iii) $\rho(\tilde{\ell}_{r,\omega}) > \rho(\ell_{r,\omega})$, if $\rho(\ell_{r,\omega}) > 1$.

PROOF: It is established, from Theorem 1, that $\ell_{r,\omega}$ and $\tilde{\ell}_{r,\omega}$ are nonnegative and irreducible matrices. Therefore, the theorem were proved following the same procedure in [1] and [6]

Corollary 1 Let $\ell_{\omega} = (I - \omega L)^{-1}[(1 - \omega)I + \omega U]$ and $\tilde{\ell}_{\omega} = (\tilde{D} - \omega\tilde{L})^{-1}[(1 - \omega)\tilde{D} + \omega\tilde{U}]$ be the AOR and preconditioned AOR iterative matrices respectively. Suppose $0 < \omega < 1$, A is an irreducible $L -$ matrix with $0 < a_{1n}a_{n1} < 1$, $0 < a_{1i}a_{i1} + a_{i,i+1}a_{i+1,i} < 1$ ($i = 2(1)n - 1$) and $0 < a_{12}a_{21} + a_{in}a_{n1} < 1$. Then

- (iv) $\rho(\tilde{\ell}_{\omega}) < \rho(\ell_{\omega})$, if $\rho(\ell_{\omega}) < 1$;
- (v) $\rho(\tilde{\ell}_{\omega}) = \rho(\ell_{\omega})$, if $\rho(\ell_{\omega}) = 1$;
- (vi) $\rho(\tilde{\ell}_{\omega}) > \rho(\ell_{\omega})$, if $\rho(\ell_{\omega}) > 1$.

RESULTS

Numerical Examples

In this section we give a numerical example to illustrate the results obtained in section 2. Thus, let consider the matrix A of (1.1), same problem suggested in [1] and [14]

Problem 1 considered by Abdullahi and Ndanusa. (2020)

$$A = \begin{pmatrix} 1 & -1/6 & -1/7-1/8 & -1/6 & -1/7 \\ -1/8 & 1 & -1/6-1/7 & -1/8 & -1/6 \\ -1/6 & -1/8 & 1 & -1/6 & -1/7 \\ -1/7 & -1/6 & -1/8 & 1 & -1/6 \\ -1/8 & -1/7 & -1/6-1/8 & 1 & -1/6 \\ -1/6 & -1/8 & -1/7-1/6 & -1/8 & 1 \end{pmatrix}$$

Problem 2 considered by Ndanusa and Adeboye. (2012)

$$A = \begin{pmatrix} 1 & -0.279-0.233 & 0 \\ -0.306 & 1 & 0 & -0.204 \\ -0.265 & 0 & 1 & -0.245 \\ 0 & -0.236-0.273 & 1 \end{pmatrix}$$

Table 1: Comparison of the spectral radii of problem 1

ω	r	$\rho(\tilde{\ell}_{r,\omega})$	$\rho(\ell_{r,\omega}(1))$	$\rho(\ell_{r,\omega}(2))$	$\rho(\ell_\omega)$
0.95	0.85	0.4718225724	0.4820339009	0.5225691676	0.6205255277
0.90	0.80	0.5147480406	0.5621098000	0.5268521180	0.6518574112
0.80	0.70	0.5917832606	0.6328652777	0.6059612644	0.7083014149
0.70	0.65	0.6516690078	0.6872375769	0.6652871216	0.7516743194
0.60	0.50	0.7211254715	0.7507664885	0.7352605039	0.8026767336
0.50	0.40	0.7769215150	0.8012067271	0.7897395742	0.8429614522
0.40	0.30	0.8282069826	0.8473247279	0.8391415217	0.8796743773
0.30	0.20	0.8756791423	0.8897969929	0.8843014020	0.9133536720
0.20	0.10	0.9198700160	0.9291414937	0.9258516890	0.9444202400
0.10	0.05	0.9605794594	0.9651811350	0.9636211849	0.9727210830

Table 2: Comparison of the spectral radii of problem 2

ω	r	$\rho(\tilde{\ell}_{r,\omega})$	$\rho(\ell_{r,\omega}(1))$	$\rho(\ell_{r,\omega}(2))$	$\rho(\ell_\omega)$
0.95	0.85	0.1919310451	0.1928998859	0.2193083711	0.3703668656
0.90	0.80	0.2468909635	0.2494857709	0.2739612623	0.4195567967
0.80	0.70	0.3489790786	0.3539827607	0.3747489815	0.5081360007
0.70	0.65	0.4373116604	0.4427761576	0.4605042387	0.5787456953
0.60	0.50	0.5331287209	0.5403654023	0.5544139067	0.6590826438
0.50	0.40	0.6182959577	0.6256102316	0.6366849191	0.7254074446
0.40	0.30	0.6999593266	0.7067677540	0.7151299118	0.7871329383
0.30	0.20	0.7786264169	0.7844051870	0.7903116697	0.8449749413
0.20	0.10	0.8546725892	0.8589468480	0.8626471382	0.8994698395
0.10	0.05	0.9278705111	0.9301085660	0.9319005606	0.9503972408

Table 1 and 2 displays the results of comparing the spectral radii of the proposed preconditioned method, for varied values of ω and r , with those of other preconditioned methods in literature. It reveals that the rate of convergence of the proposed method is faster than those of Faruk and Ndanusa (2019) and Abdullahi and Ndanusa (2020), even as all the preconditioned methods proved faster than the unpreconditioned AOR.

Conclusion

In this paper, we have proposed new preconditioner schemes for solving L -matrices linear system. The numerical examples are presented, which show the advancement on the convergence rate of AOR (SOR) preconditioned iterative schemes. Then, we have provided the comparison results among the AOR (SOR) iterative scheme and preconditioned methods for solving L -matrices linear system. Maple 19 was

used to obtain the eigenvalues and spectral radiuses.

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